THE EXPONENTIAL TIME DIFFERENCE (ETD) SCHEME FOR THE NONLINEAR SCHROEDINGER EQUATION

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Abstract

Spectral methods provide very high spatial resolution for a wide range of non-linear wave equations . In this study, the Fourth Order Runge-Kutta Exponential Time Difference (ETDRK4) method is used to numerically simulate the Gross-Pitaevskii equation. Approximate numerical solutions of the Gross-Pitaevskii equation obtained using the Matlab software . It is shown that the proposed method significantly increases the computational costs. This improvement becomes more significant, especially for large time evolutions.

Keywords: *Gross-Pitaevskii equation , splitting method, splitting , partial differential equations, approximate numerical solutions*

Introduction

Partial differential equations are a widely used mathematical apparatus in the development of models in various fields of science and technology. Unfortunately, the explicit solution of these equations in an analytical form is possible only in special simple cases, and, as a result, the possibility of analyzing mathematical models is provided by solving these equations by approximate numerical methods. In recent years, non-linear evolution equations have become a very active field for describing various areas of non-linear sciences. One-dimensional nonlinear Schrödinger equation (1D NLSE) is a classical field equation. Its most prominent applications are related to the propagation of light waves in optical fibers and planar waveguides along with many others [1]. In particular, 1D NLSE is a non-linear second-order partial differential equation applicable to both classical and quantum mechanics. The nonlinear Schrödinger equation has an extremely high universality and is used to describe wave processes in many areas of physics: in the theory of surface waves [1], in models of the evolution of plasma oscillation distributions [2], nonlinear optics [3], biophysics, etc. The non-linear Schrödinger equation describes the propagation of non-linear Langmuir waves, waves in deep water; waves in transmission lines, acoustic waves in liquids with bubbles and, above all, the propagation of optical radiation in nonlinear media. A typical application of the nonlinear Schrödinger equation is the dynamics of optical pulses in an optical fiber. The time evolution of the envelope of an optical pulse in a fiber is well approximated by the nonlinear Schrödinger equation, including the description of very long transoceanic optical communication lines, see, for example, [4, 5].

The nonlinear Schrödinger equation under consideration is a nonlinear differential equation with partial derivatives, which in the general case cannot be solved analytically. Therefore, numerical simulation methods are used to solve this problem. In this section, we present the exponential time difference (ETD) scheme that was used in [6] and [7] to numerically solve the Schrödinger equation in semiclassical mode. Finally, note that ETD is a very powerful general-purpose numerical method for solving GPE that can be applied to a large number of different physical situations [8]. The efficiency of this method and the high accuracy of the solutions make ETD a good choice for solving experimental situations that are very demanding on numerical characteristics [9].

The idea of ETD methods is similar to the integrating factor method (see, for example, [10] or [11]): we multiply both sides of the differential equation by some integrating factor, then we make a change of variable that allows us to solve the linear part exactly, and finally, we use the numerical method of our choice to solve the transformed non-linear part [12].

Finally, note that ETD is a very powerful general-purpose numerical method for solving GPE that can be applied to a wide variety of physical situations. The efficiency of this method and the high accuracy of the solutions make ETD a good choice for solving experimental situations that are very numerically demanding.

Two-dimensional nonlinear Schrödinger equation

Considering the good behavior of our simulation for a one-dimensional problem and the fact that MATLAB © also implements multidimensional discrete Fourier transforms and their inverses (with two variables), we thought that slight modifications to our program would allow us to model a two-dimensional cubic non-linear Schrödinger equation [13]

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2}+q\,|\,\psi|^2\psi=0,\quad (x,y)\in\mathbf{R}^2,\quad t>0. \tag{3}$$

Where q is a real constant and a sign q gives two very different problems: the focus case if $q>0$ and the defocus case for $q<0$. This mathematical model is found in many physical applications; this is especially important in optics (for more details, see [13]). In our first test with $q=1$ and initial condition

$$\psi_0(x,y)=2+0.01\cdot\sin(x+\pi/4)\sin(y+\pi/4).$$

$$(x,y)\in[-\pi,\ \pi]\times[-\pi,\ \pi].$$

c $\|\psi_0\|\approx 12.56$, using $(256)^2$ collocation points, we got Fig. 1-2.

Results and discussions

Computer time takes about 64.30 seconds. In Fig. 1-2 we have plotted the time evolution for $q=1$, s $\|\psi_0\|_2\approx 8.8623$. We can see that the dispersion process is dominant. Figure 1 shows partial output of the first program solution of NLSE when the discretization size $\Delta x=0.1$, spatial domain is $-\pi\le x,y\le\pi$ and step size is on the interval $0\le t\le 10$. Figure 2 shows partial output of the first program solution of CSE when the discretization size $\Delta x=0.3$, spatial domain is $-\pi\le x,y\le\pi$ and step size is on the interval $0\le t\le 10$.

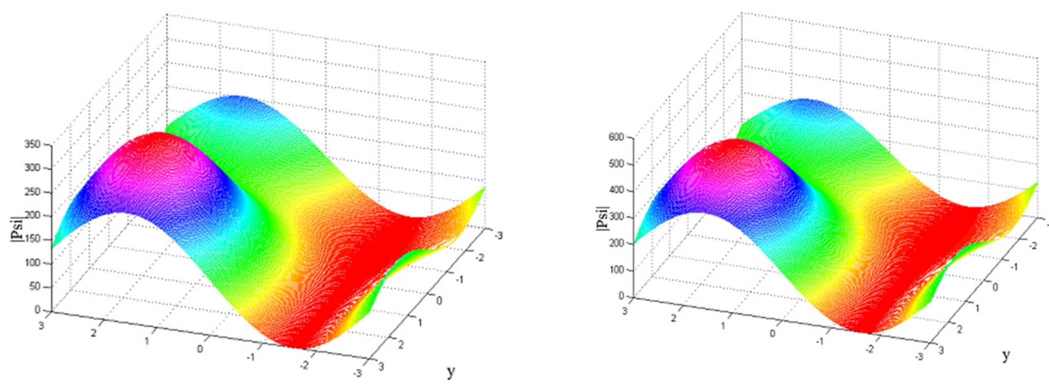

Fig.1. The solution of NLSE when the discretization size $\Delta x=0.1$, spatial domain is $-\pi\le x,y\le\pi$ and step size is on the interval $0\le t\le 10$.

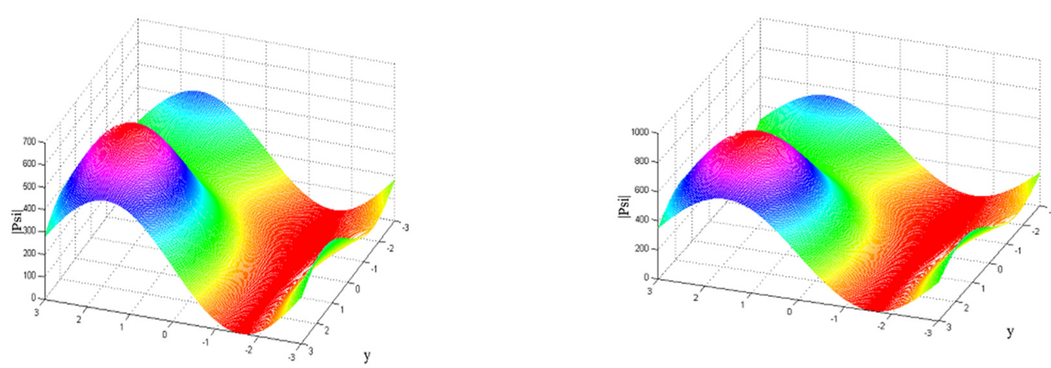

Fig.2. The solution of NLSE when the discretization size $\Delta x=0.3$, spatial domain is $-\pi\le x,y\le\pi$ and step size is on the interval $0\le t\le 10$.

According to the results presented in these figures, this method provides a high accuracy of the numerical solution of the Gross-Pitaevskii equation . On the other hand, as can be seen from the figures, the result obtained by the implicit exponential difference scheme has better results than the results obtained by other numerical schemes. These calculations show that the accuracy of the solutions is quite high even in the case of a small number of grid nodes. Finally, note that ETD is a very powerful general-purpose numerical method for solving GPE that can be applied to a wide variety of physical situations. The efficiency of this method and the high accuracy of the solutions make ETD a good choice for solving experimental situations that are very numerically demanding.

CONCLUSION

We concluded that they are more accurate than other methods (standard factor integration methods or linear-implicit schemes); they have good stability properties and are widely applicable to non-linear wave equations. We calculated and studied the numerical stability function of the ETDRK 4 methods and, in addition to their good stability properties, identified the reasons for their good behavior for dissipative and dispersion problems.

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